

Critical temperature for the ferromagnetic transition in the Ising model on multiplex networks

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Abstract

Multiplex networks consist of a fixed set of nodes connected by several sets of edges which are generated separately and correspond to different networks ("layers"). Here, the Ising model on multiplex networks with two layers is considered, with spins located in the nodes and edges corresponding to ferromagnetic interactions between them. Critical temperatures for the ferromagnetic transition are evaluated for the layers in the form of random Erdős-Rényi graphs or heterogeneous scale-free networks using the mean-field approximation and the replica method, from the replica symmetric solution. Both analytic methods require the use of different "partial" order parameters or magnetizations, respectively, associated with different layers of the multiplex network, and yield qualitatively similar results. If the exchange integrals corresponding to different layers are equal, in the case of random Erdős-Rényi layers the critical temperature for the Ising model on a multiplex network is equal to that for the Ising model on a network being a superposition of the two layers, while in the case when at least one layer is a strongly heterogeneous network the critical temperatures in these two cases differ noticeably. Besides, the critical temperature depends sensitively on the correlations between the degrees of nodes within different layers. The critical temperature evaluated from the replica symmetric solution shows satisfactory quantitative agreement with that obtained from Monte Carlo simulations.

Keywords: multiplex networks; phase transitions; Ising model; mean-field theory; replica method.

1. Introduction

Due to the ubiquity and importance of complex networks in many areas of social life, science and technology much research has been devoted to this topic in the last decades [1, 2]. Also the physics of interacting systems on complex networks is a rapidly developing branch of statistical physics [3, 4]. Among the latter systems the ferromagnetic (FM) and spin-glass (SG) transitions in the Ising model was investigated on various complex networks, including heterogeneous scale-free (SF) networks, e.g., by means of heterogeneous mean-field (MF) theory [6, 7, 8], the replica method [7, 9], belief propagation algorithm [10, 11] and Monte Carlo (MC) simulations [12, 13, 14, 15]. In the context of recent interest in even more complex structures ("networks of networks") much attention has been devoted to multiplex networks (MNs) which consist of a fixed set of nodes connected by various sets of edges called layers [16, 17, 18]. MNs naturally emerge in various social systems (e.g., transportation or communications networks), and interacting systems on such structures exhibit rich variety of collective behaviours and critical phenomena. For example, percolation transition [19, 20, 21], cascading failures [22], diffusion processes [23, 24], epidemic spreading [25, 26], etc., were studied on MNs. In particular, investigation of the Ashkin-Teller model on a MN, treated as a model for interacting systems between two species of Ising spins placed on two layers, revealed rich critical behaviour including occurrence of continuous, discontinuous and mixed-order phase transitions, depending on the parameters of the model [27].

In this paper the Ising model on MNs is investigated, with spins placed on a fixed set of nodes and with separately generated sets of edges (layers) corresponding to (in general, different) FM exchange interactions; the layers can have, e.g., a structure of random Erdős-Rényi (ER) graphs or heterogeneous SF networks. In contrast with the Ashkin-Teller model on MNs [27] in the Ising model in the absence of the external field only continuous FM transition is expected to occur. The aim of this paper is to evaluate the critical temperature for the FM transition in the Ising model on MNs with different kinds of layers. In Sec. 2 the model is defined. In Sec. 3 heterogeneous MF theory for the model is formulated and the critical temperature in the MF approximation is obtained. In Sec. 4 a more rigorous approach based on the replica method known from the SG theory is developed and again applied to evaluate the critical temperature for the FM transition. In both approaches analytic calculations are performed for MNs with layers in the form of ran-

dom ER graphs and heterogeneous SF networks. The main results of Sec. 3 and Sec. 4 are that in the case of layers with high density of connections the critical temperatures obtained from the heterogeneous MF theory and the replica method are close to each other and show qualitative agreement with results of MC simulations; better quantitative agreement is obtained in the latter case. Moreover, even if the exchange integrals in all layers are equal, despite the apparent simplicity of the model, the critical temperature for the Ising model on a MN with heterogeneous SF layers can differ from that for the Ising model on a network being a superposition of the layers (called henceforth a super-network). In Sec. 5, summary and conclusions are presented.

2. The model

MNs consist of a fixed set of nodes connected by several sets of edges; the set of nodes with each set of edges forms a network which is called a layer of a MN [17, 18]. In the following, for simplicity, MNs with N nodes and only two layers denoted as $G^{(A)}$, $G^{(B)}$ are considered. The layers (strictly speaking, the sets of edges within each layer) are generated separately, and, in most cases considered in this paper, independently. As a result, multiple connections between nodes are not allowed within the same layer, but the same nodes can be connected by multiple edges belonging to different layers. The nodes $i = 1, 2, \dots, N$ are characterized by their degrees $k_i^{(A)}$, $k_i^{(B)}$ within each layer, i.e., the number of edges attached to them within each layer. The, possibly heterogeneous, distributions of the degrees of nodes within each layer are denoted as $p_{k^{(A)}}$, $p_{k^{(B)}}$; in the case of independently generated layers the joint distribution of the degrees of nodes in the MN is $p_{k^{(A)}, k^{(B)}} = p_{k^{(A)}} p_{k^{(B)}}$. The mean degree of the nodes and the second moment of the distribution of the degrees of nodes within each layer are denoted as $\langle k^{(A)} \rangle$ ($\langle k^{(B)} \rangle$) and $\langle k^{(A)2} \rangle$ ($\langle k^{(B)2} \rangle$) for the layer $G^{(A)}$ ($G^{(B)}$), respectively. In this paper only fully overlapping MNs are considered, with all N nodes belonging to both layers; the case of partly overlapping MNs, with only a fraction of nodes belonging to both layers, is left for future research.

In the FM Ising model on a MN with two layers two-state spins $s_i = \pm 1$ are located in the nodes $i = 1, 2, \dots, N$ and edges within the layers $G^{(A)}$, $G^{(B)}$ correspond to exchange interactions with integrals $J^{(A)} > 0$, $J^{(B)} > 0$,

respectively. The Hamiltonian of the model is

$$H = -J^{(A)} \sum_{(i,j) \in G^{(A)}} s_i s_j - J^{(B)} \sum_{(i,j) \in G^{(B)}} s_i s_j, \quad (1)$$

where the sums are over all edges belonging to the layer $G^{(A)}$ ($G^{(B)}$). Thus the local field acting on the spin in the node i is

$$I_i = J^{(A)} \sum_{\{j:(i,j) \in G^{(A)}\}} s_j + J^{(B)} \sum_{\{j:(i,j) \in G^{(B)}\}} s_j, \quad (2)$$

where the sums are over all nodes j connected to the node i by edges within the layer $G^{(A)}$ ($G^{(B)}$). It should be emphasised that in the model under study there is only one spin s_i located in each node which interacts with all its neighbours within all layers. This is in contrast with the related Ashkin-Teller model on a MN with two layers considered in Ref. [27], where spins interacting via exchange interactions within each layer are different, and different spins located in the same node interact via additional four-spin interactions.

The space of parameters of the model is large and comprises $J^{(A)}$, $J^{(B)}$ and characteristics of the distributions $p_{k^{(A)}}$, $p_{k^{(B)}}$. Thus, for simplicity, in the following only the Ising model on a MN with $J^{(A)} = J^{(B)} = J > 0$ will be considered. At a first glance this case seems trivial since the Hamiltonian, Eq. (1), is then identical with that for the Ising model on a super-network being a superposition of the two layers. However, these two models are not equivalent because in the model on a MN certain probabilities (e.g., the probability that a node is linked by an edge to a node of a given degree) or statistical averages (e.g., over different realizations of the set of edges) should be evaluated separately for the two layers. In the case of heterogeneous layers this leads to noticeable differences in the critical temperature for the FM transition for the two models.

3. The mean-field approach

It is known that first, and in many cases even quantitatively correct approximation for the critical temperature for the FM transition in the Ising model on heterogeneous networks can be obtained from the heterogeneous MF theory [6, 7]. Also in the case of MNs the MF approach was successfully applied, e.g., in the studies of epidemic spreading [25, 26]. Hence, in this

section appropriate theory is developed for the Ising model on a MN with two separately generated, possibly different layers; examples of such MNs are discussed in Sec. 3.1. For simplicity only the case of independently generated layers is considered, thus it is possible to assume $p_{k^{(A)},k^{(B)}} = p_{k^{(A)}}p_{k^{(B)}}$ in the calculations. The heterogeneous MF theory for systems on MNs differs from that for systems on networks since the probabilities that a node is connected to a node with a given degree must be evaluated separately for each layer. As a result, two magnetization-like "partial" order parameters, related to the two layers, are necessary to characterize the FM transition in the Ising model, which is shown in Sec. 3.2. General results for the critical temperature in the case of heterogeneous layers are derived in Sec. 3.3, and results for the particular cases of random ER and SF layers are presented in Sec. 3.4. Finally, in Sec. 3.5 analytic results are compared with those from MC simulations.

3.1. The network models

The simplest way to generate a MN with independent layers and with given distributions of the degrees of nodes within layers is probably to use the Configuration Model [28] separately and independently for each layer. This method is particularly useful for generation of heterogeneous SF layers. To generate the first layer $G^{(A)}$, the algorithm starts with assigning to each node i , in a set of N nodes, a degree, i.e., a random number $k_i^{(A)}$ of ends of edges drawn from a given probability distribution $p_{k^{(A)}}$, with $\tilde{m} < k_i^{(A)} < N$ (the minimum degree of node is \tilde{m} , and the maximum one $N - 1$), with the condition that the sum $\sum_i k_i^{(A)}$ is even. The layer is completed by connecting pairs of the ends of edges chosen uniformly at random to make complete edges, respecting the preassigned sequence $k_i^{(A)}$ and under the condition that multiple and self-connections are forbidden. Consecutive layers are generated in a similar way, with the degrees assigned randomly to the nodes from the possibly different probability distributions. Instead, random ER layers can be generated by selecting randomly and with uniform probability $N\langle k \rangle/2$ pairs of nodes and linking them with edges, where $\langle k \rangle$ is the desired mean degree of nodes within the layer [29]. Using these methods, MNs with layers with different structure and statistical properties can be easily obtained.

3.2. Heterogeneous mean field theory

There are different ways to derive the MF equations for the order parameter (in general, a sort of magnetization) for the Ising model on, possibly

heterogeneous, networks [6, 7]. Here we adopt the approach based on the Master equation for the probability that at time t the system is in the spin configuration (s_1, s_2, \dots, s_N) , $s_i = \pm 1$,

$$\begin{aligned} \frac{d}{dt} P(s_1, s_2, \dots, s_N; t) = & - \sum_{j=1}^N w_j(s_j) P(s_1, s_2, \dots, s_j, \dots, s_N; t) \\ & + \sum_{j=1}^N w_j(-s_j) P(s_1, s_2, \dots, -s_j, \dots, s_N; t), \end{aligned} \quad (3)$$

where $w_i(s_i)$ is the transition rate between two spin configurations which differ by a single flip of one spin, e.g., that in the node i . For example, let us assume that the system obeys the Glauber dynamics (used in MC simulations in Sec. 3.5) with

$$w_i(s_i) = \frac{1}{2} [1 - s_i \tanh(\beta I_i)], \quad (4)$$

where $\beta = 1/T$. Then, multiplying both sides of Eq. (3) by s_i and performing an ensemble average it is obtained that

$$\frac{d\langle s_i \rangle}{dt} = -\langle s_i \rangle + \langle \tanh(\beta I_i) \rangle. \quad (5)$$

The mean-field approximation consists in replacing in Eq. (2)

$$I_i \rightarrow \langle I_i \rangle = J \sum_{\{j:(i,j) \in G^{(A)}\}} \langle s_j \rangle + J \sum_{\{j:(i,j) \in G^{(B)}\}} \langle s_j \rangle, \quad (6)$$

so that

$$\frac{d\langle s_i \rangle}{dt} = -\langle s_i \rangle + \tanh(\beta \langle I_i \rangle). \quad (7)$$

The basic assumption of the heterogeneous MF theory for the Ising model on networks is that the nodes of the network are divided into classes according to their degrees and that the average values of spins in nodes belonging to the same class are equal. In the case of MNs the division into classes should be performed with respect to the degrees of nodes within each layer. Thus, for a MN consisting of two layers $G^{(A)}$, $G^{(B)}$, the nodes are divided into classes according to their degrees $(k^{(A)}, k^{(B)})$ and it is assumed that the average values of spins in nodes belonging to each such class is equal to $\langle s_{k^{(A)}, k^{(B)}} \rangle$. For independent layers the probability that the edge of the layer

$G^{(A)}$ attached at one end to the node i is linked at the other end to the node with degrees $(k^{(A)}, k^{(B)})$ is

$$\frac{p_{k^{(A)}} p_{k^{(B)}} k^{(A)}}{\sum_{k^{(A)}, k^{(B)}} p_{k^{(A)}} p_{k^{(B)}} k^{(A)}} = \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(A)}}{\langle k^{(A)} \rangle}, \quad (8)$$

and similarly for the layer $G^{(B)}$. Thus, the number of nodes with degrees $(k^{(A)}, k^{(B)})$ connected to the node i by edges of the layer $G^{(A)}$ is

$$k_i^{(A)} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(A)}}{\langle k^{(A)} \rangle}, \quad (9)$$

and similarly for the layer $G^{(B)}$. Hence, replacing the sums over the indices of nodes by sums over the classes of nodes, Eq. (6) and (7) can be written as

$$\begin{aligned} \langle I_i \rangle &= J k_i^{(A)} \sum_{k^{(A)}, k^{(B)}} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(A)}}{\langle k^{(A)} \rangle} \langle s_{k^{(A)}, k^{(B)}} \rangle \\ &\quad + J k_i^{(B)} \sum_{k^{(A)}, k^{(B)}} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(B)}}{\langle k^{(B)} \rangle} \langle s_{k^{(A)}, k^{(B)}} \rangle \\ &= J \left(k_i^{(A)} \langle S^{(A)} \rangle + k_i^{(B)} \langle S^{(B)} \rangle \right), \end{aligned} \quad (10)$$

$$\frac{d\langle s_i \rangle}{dt} = -\langle s_i \rangle + \tanh \left[J\beta \left(k_i^{(A)} \langle S^{(A)} \rangle + k_i^{(B)} \langle S^{(B)} \rangle \right) \right], \quad (11)$$

where the following quantities, referred to as "partial" order parameters, were introduced,

$$\begin{aligned} \langle S^{(A)} \rangle &\equiv \frac{1}{N \langle k^{(A)} \rangle} \sum_{i=1}^N k_i^{(A)} \langle s_i \rangle = \sum_{k^{(A)}, k^{(B)}} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(A)}}{\langle k^{(A)} \rangle} \langle s_{k^{(A)}, k^{(B)}} \rangle \\ \langle S^{(B)} \rangle &\equiv \frac{1}{N \langle k^{(B)} \rangle} \sum_{i=1}^N k_i^{(B)} \langle s_i \rangle = \sum_{k^{(A)}, k^{(B)}} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(B)}}{\langle k^{(B)} \rangle} \langle s_{k^{(A)}, k^{(B)}} \rangle. \end{aligned} \quad (12)$$

Multiplying both sides of Eq. (11), by $\frac{k_i^{(A)}}{N \langle k^{(A)} \rangle}$ ($\frac{k_i^{(B)}}{N \langle k^{(B)} \rangle}$), performing the sum over the nodes and replacing it with the sum over the classes of nodes results in the following system of MF equations for the "partial" order parameters

$$\frac{d\langle S^{(A)} \rangle}{dt} = -\langle S^{(A)} \rangle + \sum_{k^{(A)}, k^{(B)}} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(A)}}{\langle k^{(A)} \rangle} \tanh \left[J\beta \left(k_i^{(A)} \langle S^{(A)} \rangle + k_i^{(B)} \langle S^{(B)} \rangle \right) \right]$$

$$\frac{d\langle S^{(B)} \rangle}{dt} = -\langle S^{(B)} \rangle + \sum_{k^{(A)}, k^{(B)}} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(B)}}{\langle k^{(B)} \rangle} \tanh \left[J\beta \left(k_i^{(A)} \langle S^{(A)} \rangle + k_i^{(B)} \langle S^{(B)} \rangle \right) \right]. \quad (13)$$

It should be noted that equations similar to Eq. (13) were obtained for the Ising model on a modular network consisting of two heterogeneous networks (modules), with the density of connections, corresponding to the exchange interactions, within each module much higher than that between the modules [8, 30]. The difference is that in the case of modular networks the "partial" order parameters $\langle S^{(A)} \rangle$, $\langle S^{(B)} \rangle$ are obtained by summing weighted average values of spins within each module (i.e., the summations are performed over two separate sets of nodes), while in Eq. (12) both summations are over the same set of N nodes. A certain degree of separation of nodes in the sums in Eq. (12) can be achieved in partly overlapping MNs; investigation of this case is beyond the scope of this paper.

3.3. General equations for the critical temperature

Eq. (13) has a fixed point $(\langle S^{(A)} \rangle, \langle S^{(B)} \rangle) = (0, 0)$ corresponding to the paramagnetic phase. Expanding Eq. (13) in the vicinity of this fixed point up to linear terms yields

$$\begin{aligned} \frac{d\langle S^{(A)} \rangle}{dt} &= \left(-1 + \beta J \frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} \right) \langle S^{(A)} \rangle + \beta J \langle k^{(B)} \rangle \langle S^{(B)} \rangle \\ \frac{d\langle S^{(B)} \rangle}{dt} &= \beta J \langle k^{(A)} \rangle \langle S^{(A)} \rangle + \left(-1 + \beta J \frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} \right) \langle S^{(B)} \rangle. \end{aligned} \quad (14)$$

The paramagnetic fixed point becomes unstable, and the FM phase occurs, if one of the eigenvalues of Eq. (14) crosses zero which takes place if

$$\begin{aligned} \beta^2 J^2 \left(\frac{\langle k^{(A)2} \rangle \langle k^{(B)2} \rangle}{\langle k^{(A)} \rangle \langle k^{(B)} \rangle} - \langle k^{(A)} \rangle \langle k^{(B)} \rangle \right) \\ - \beta J \left(\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} + \frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} \right) + 1 = 0. \end{aligned} \quad (15)$$

In general, Eq. (15) has two solutions with

$$T_{c\pm} = 2J \frac{\frac{\langle k^{(A)2} \rangle \langle k^{(B)2} \rangle}{\langle k^{(A)} \rangle \langle k^{(B)} \rangle} - \langle k^{(A)} \rangle \langle k^{(B)} \rangle}{\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} + \frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} \pm \sqrt{\Delta}}, \quad (16)$$

with

$$\Delta = \left(\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} - \frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} \right)^2 + 4\langle k^{(A)} \rangle \langle k^{(B)} \rangle.$$

The higher temperature T_{c-} corresponds to the critical temperature for the FM transition, $T_c^{MF} = T_{c-}$. Below T_c^{MF} the paramagnetic state is unstable, and the instability at $T = T_{c+} < T_c^{MF}$ has no physical meaning.

If the distributions of the degrees of nodes within each layer $G^{(A)}$, $G^{(B)}$ are identical, $p_{k^{(A)}} = p_{k^{(B)}} = p_k$, and thus $\langle k^{(A)} \rangle = \langle k^{(B)} \rangle = \langle k \rangle$, $\langle k^{(A)2} \rangle = \langle k^{(B)2} \rangle = \langle k^2 \rangle$, then

$$T_{c\pm} = J \left(\frac{\langle k^2 \rangle}{\langle k \rangle} \mp \langle k \rangle \right). \quad (17)$$

It can be easily verified that in this case the eigenvalue of Eq. (14) which crosses zero at $T = T_c = T_{c-}$ corresponds to the eigenvector with $\langle S^{(A)} \rangle = \langle S^{(B)} \rangle$, i.e., in fact to the FM phase.

Similar results were obtained for the Ising model on modular networks [8, 30], where also two values of the critical temperature were found. The higher value corresponds to the usual FM transition, in which all spins in both modules tend to align in parallel. The lower value corresponds to the occurrence of another ordered state, in which spins within each module tend to align in parallel, but antiparallel to the spins in the other module. In the case of small density of connections between spins in different modules this state is a long-living metastable state, and transition from this state to a stable FM state as the temperature increases can be discontinuous, though only FM interactions between spins are present [30]. Similar phenomenon can, in principle, occur in the Ising model on partly overlapping MNs with only a small fraction of nodes belonging to each layer; investigation of this case is beyond the scope of this paper.

3.4. Special cases

3.4.1. Random Erdős-Rényi layers

If the two layers $G^{(A)}$, $G^{(B)}$ are independently generated random ER graphs with mean degrees of nodes $\langle k^{(A)} \rangle$, $\langle k^{(B)} \rangle$, respectively, for large N the distributions of the degrees of nodes can be assumed as Poisson ones, $p_{k^{(A)}} = \frac{\langle k^{(A)} \rangle^{k^{(A)}}}{k^{(A)}!} e^{-\langle k^{(A)} \rangle}$, $p_{k^{(B)}} = \frac{\langle k^{(B)} \rangle^{k^{(B)}}}{k^{(B)}!} e^{-\langle k^{(B)} \rangle}$, for which $\langle k^{(A)2} \rangle = \langle k^{(A)} \rangle + \langle k^{(A)} \rangle^2$, $\langle k^{(B)2} \rangle = \langle k^{(B)} \rangle + \langle k^{(B)} \rangle^2$. Then the critical temperature, Eq. (16), is

$$T_c^{MF} = T_{c-} = J \left(1 + \langle k^{(A)} \rangle + \langle k^{(B)} \rangle \right). \quad (18)$$

Taking into account the way of generation of random ER layers it is easy to see that the resulting MN is also a random ER graph with the mean degree of nodes $\langle k \rangle = \langle k^{(A)} \rangle + \langle k^{(B)} \rangle$. Thus, it is fully equivalent to the super-network being a superposition of the two layers, with the degrees of nodes $k_i = k_i^{(A)} + k_i^{(B)}$ obeying the Poisson distribution $p_k = \frac{\langle k \rangle^k}{k!} e^{-\langle k \rangle}$ with $\langle k \rangle = \langle k^{(A)} \rangle + \langle k^{(B)} \rangle$. Hence, critical temperature for the FM transition for the Ising model on the super-network is

$$T_c^{sup} = J \langle k^2 \rangle / \langle k \rangle = J (1 + \langle k \rangle) = J (1 + \langle k^{(A)} \rangle + \langle k^{(B)} \rangle) = T_c^{FM}$$

and the Ising model on the MN and the corresponding super-network are equivalent.

3.4.2. Scale-free layers

If the two layers $G^{(A)}$, $G^{(B)}$ are independently generated SF networks with $p_{k^{(A)}} = (\gamma^{(A)} - 1) \tilde{m}^{\gamma^{(A)}-1} (k^{(A)})^{-\gamma^{(A)}}$, $p_{k^{(B)}} = (\gamma^{(B)} - 1) \tilde{m}^{\gamma^{(B)}-1} (k^{(B)})^{-\gamma^{(B)}}$, with the same minimum node degree \tilde{m} and with the exponents in the power scaling laws of the distributions of the degrees of nodes $\gamma^{(A)} > 3$, $\gamma^{(B)} > 3$, the critical temperature, Eq. (16), is

$$T_c^{MF} = T_{c-} = 2J\tilde{m} \frac{\frac{\gamma^{(A)}-2}{\gamma^{(A)}-3} \frac{\gamma^{(B)}-2}{\gamma^{(B)}-3} - \frac{\gamma^{(A)}-1}{\gamma^{(A)}-2} \frac{\gamma^{(B)}-1}{\gamma^{(B)}-2}}{\frac{\gamma^{(A)}-2}{\gamma^{(A)}-3} + \frac{\gamma^{(B)}-2}{\gamma^{(B)}-3} - \sqrt{\Delta}}, \quad (19)$$

with

$$\Delta = \left(\frac{\gamma^{(A)}-2}{\gamma^{(A)}-3} - \frac{\gamma^{(B)}-2}{\gamma^{(B)}-3} \right)^2 + 4 \frac{\gamma^{(A)}-1}{\gamma^{(A)}-2} \frac{\gamma^{(B)}-1}{\gamma^{(B)}-2}.$$

If for at least one layer, say $G^{(A)}$, the distribution of the degrees of nodes has for large N a diverging second moment, $\langle k^{(A)2} \rangle \xrightarrow{N \rightarrow \infty} \infty$, which occurs for SF layers with $2 < \gamma^{(A)} \leq 3$, then expanding in Eq. (16)

$$\sqrt{\Delta} \approx \frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} \left(1 - \frac{\langle k^{(A)} \rangle}{\langle k^{(A)2} \rangle} \frac{\langle k^{(B)} \rangle}{\langle k^{(B)2} \rangle} + \dots \right)$$

yields

$$T_c^{MF} = T_{c-} \approx J \frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} \xrightarrow{N \rightarrow \infty} \infty,$$

i.e., there is only crossover (dependent on N) temperature for the FM transition and in the thermodynamic limit the system remains in the FM phase

at any temperature. This is the same situation as for the Ising model on any network with a divergent second moment of the distribution of the degrees of nodes [7, 10].

It is interesting to compare the result of Eq. (19) with the critical temperature for the Ising model on a super-network in which the degree of each node is a sum of independent degrees of this node within each layer, $k_i = k_i^{(A)} + k_i^{(B)}$. The distribution of the degrees of nodes for the super-network is

$$p_k = (\gamma^{(A)} - 1) \tilde{m}^{\gamma^{(A)}-1} (\gamma^{(B)} - 1) \tilde{m}^{\gamma^{(B)}-1} \int_{\tilde{m}}^{k-\tilde{m}} (k^{(A)})^{-\gamma^{(A)}} (k - k^{(A)})^{-\gamma^{(B)}} dk^{(A)}. \quad (20)$$

The analytic form of p_k is complex [31, 32], but the distribution can be obtained by evaluating the integral in Eq. (20) numerically. In particular, up to leading term, $p_k \propto k^{-\gamma}$ for $k \geq 2\tilde{m}$, where $\gamma = \min\{\gamma^{(A)}, \gamma^{(B)}\}$. Then the critical temperature for the Ising model on the super-network is [7]

$$T_c^{sup} = J \frac{\langle k^2 \rangle}{\langle k \rangle}, \quad (21)$$

where the averages are taken over p_k . However, even though exchange integrals within each layer are equal, the Ising models on a MN with SF layers and on a corresponding super-network are not equivalent. The general formula of Eq. (21) is obtained from the heterogeneous MF theory for the Ising model on complex networks under the assumption that the probability that the edge attached at one end to the node i is linked at the other end to the node with degree k is $p_k k / \langle k \rangle$. In contrast, in the case of a MN with independently generated layers such probabilities should be evaluated separately for each layer and are given by Eq. (8). As a result, noticeable differences between the critical temperatures $T_c^{MF} = T_{c-}$ given by Eq. (16) and T_c^{sup} given by Eq. (21) for the two above-mentioned Ising models appear (see Sec. 3.5).

3.5. Comparison with Monte Carlo simulations

The Ising model was investigated numerically on MNs with two layers in the form of independent SF networks with $J = 1$, different parameters $\gamma^{(A)}$, $\gamma^{(B)}$, \tilde{m} and numbers of nodes N , generated from the Configuration Model (Sec. 3.1). MC simulations were performed using Glauber dynamics with the single spin-flip probability $w_i(s_i)$ given by Eq. (4). The critical temperature

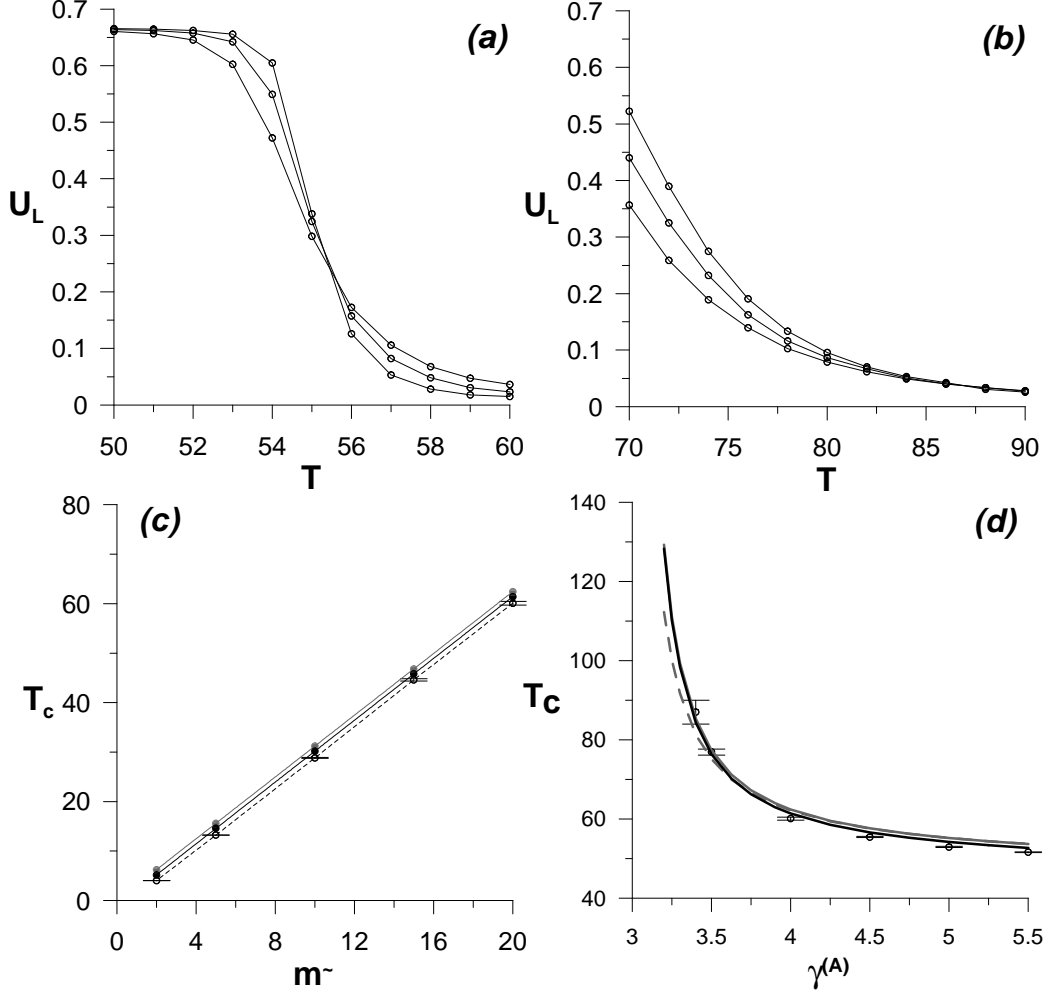


Figure 1: (a) Binder cumulants U_L vs. T for the Ising model on MNs with SF layers with $J = 1$, $\gamma^{(A)} = 4.5$, $\gamma^{(B)} = 5.5$ and (from bottom to top for small T) $N = 5000$, 10000, 20000; (b) as in (a), for $\gamma^{(A)} = 3.4$; (c) Critical temperature vs. \tilde{m} for the Ising model on MNs with SF layers with $J = 1$, $\gamma^{(A)} = 4.0$, $\gamma^{(B)} = 5.5$: T_c^{MC} from MC simulations (circles) and linear least-squares fit $T_c^{MC} = 3.12\tilde{m}$ (dotted line), T_c^{MF} from the heterogeneous MF theory, Eq. (19) (gray dots) and linear least-squares fit $T_c^{MF} = 3.12\tilde{m}$ (gray solid line), T_c^{RS} from the RS solution, Eq. (45) (black dots) and linear least-squares fit $T_c^{RS} = 3.12\tilde{m}$ (black solid line); (d) Critical temperature vs. $\gamma^{(A)}$ for the Ising model on MNs with SF layers with $J = 1$, $\gamma^{(B)} = 5.5$, $\tilde{m} = 20$: T_c^{MC} from MC simulations (circles), T_c^{MF} from the heterogeneous MF theory, Eq. (19) (gray solid line), T_c^{sup} for the Ising model on a super-network, from the heterogeneous MF theory, Eq. (21) (gray dotted line), T_c^{RS} from the RS solution, Eq. (45) (black solid line).

for the FM transition T_c^{MC} was determined from the intersection point of the Binder cumulants U_L vs. T for different N [33],

$$U_L = \left[1 - \frac{\langle M^4 \rangle_t}{3 \langle M^2 \rangle_t^2} \right]_{av}, \quad (22)$$

where $M = N^{-1} \sum_{i=1}^N s_i$ is the usual magnetization, $\langle \cdot \rangle_t$ denotes the time average for the simulation of the Ising model on a particular MN, and $[\cdot]_{av}$ denotes the average over random realizations of the MN with given N and distributions of the degrees of nodes $p_{k^{(A)}}, p_{k^{(B)}}$. Exemplary curves U_L vs. T are shown in Fig. 1(a,b) for fixed $\gamma^{(B)} = 5.5$ and different $\gamma^{(A)}$. For $\gamma^{(A)} \geq 3.5$ the intersection point of the cumulants for different N can be determined with high accuracy (Fig. 1(a)). For $\gamma^{(A)} < 3.5$ the intersection, if any, occurs in the region of temperatures where the curves are flat and close to zero, and T_c^{MC} is either determined with relatively high error or cannot be determined at all (Fig. 1(b)).

Comparison of the critical temperatures obtained from the MC simulations and from the heterogeneous MF theory, Eq. (19), for the Ising model on MNs with SF layers with $J = 1$ and with fixed $\gamma^{(A)} > 3$, $\gamma^{(B)} > 3$ and different \tilde{m} (Fig. 1(c)) shows that T_c^{MF} is systematically higher than T_c^{MC} . The first source of this discrepancy is the approximate MF character of Eq. (19) as well as the more general Eq. (16). The second one originates from the fact that in the analytic calculations leading to Eq. (19) the distributions $p_{k^{(A)}}, p_{k^{(B)}}$ of the degrees of nodes within the layers were assumed continuous, with $k^{(A)} \geq \tilde{m}$, $k^{(B)} \geq \tilde{m}$, while in the layers generated using the Configuration Model only integer values of the degrees of nodes are allowed and there are upper constraints for the maximum degrees, $k^{(A)} < N$, $k^{(B)} < N$. Nevertheless, the relative difference between T_c^{MF} and T_c^{MC} is not large and decreases with \tilde{m} , and thus with $\langle k^{(A)} \rangle$, $\langle k^{(B)} \rangle$, as expected for the MF theory. Moreover, both critical temperatures show the same linear dependence on \tilde{m} predicted by Eq. (19). These results are qualitatively similar to those obtained from the heterogeneous MF theory and MC simulations of the Ising model on SF networks [13].

In Fig. 1(d) the critical temperatures for the Ising model on MNs with SF layers with $J = 1$, fixed $\gamma^{(B)} = 5.5$ and high $\tilde{m} = 20$ and with different $\gamma^{(A)} > 3$ are shown. It can be seen that T_c^{MF} diverges as $\gamma^{(A)} \rightarrow 3$, and thus $\langle k^{(A)2} \rangle \rightarrow \infty$, as expected. For $\gamma^{(A)} \geq 4$ the MF critical temperature overestimates that obtained from MC simulations, as discussed above. On

the other hand, for $\gamma^{(A)} \leq 3.5$ the diverging T_c^{MF} shows even quantitative agreement with T_c^{MC} ; unfortunately, as mentioned above, for $\gamma^{(A)} \rightarrow 3$ the critical temperature cannot be accurately determined from MC simulations and comparison between numerical and analytic results is dubious.

In Fig. 1(d) the critical temperatures T_c^{sup} , Eq. (21), are also shown for the Ising model on a super-network being a superposition of two SF layers with $J = 1$, fixed $\gamma^{(B)} = 5.5$ and high $\tilde{m} = 20$ and with different $\gamma^{(A)} < 3$. It can be seen that T_c^{sup} is almost equal to T_c^{MF} for $\gamma^{(A)} \approx \gamma^{(B)}$ (in fact, for $\gamma^{(A)} = \gamma^{(B)}$ there is $T_c^{sup} = T_c^{MF}$, and the value of the MF critical temperature is given by Eq. (17)). However, T_c^{sup} deviates from T_c^{MF} (is systematically smaller) as $\gamma^{(A)} \rightarrow 3$; as a result, T_c^{sup} in this range of parameters underestimates the critical temperature T_c^{MC} obtained from MC simulations. This shows that the Ising models on a MN with SF layers and on a corresponding super-network being a superposition of the two SF layers are not equivalent, and a noticeable difference between the MF critical temperatures of the two models can appear. In fact, similar difference was observed if only one layer was a SF network, and the other one was, e.g., a random ER or random regular graph.

4. The approach using the replica method

In this section the FM Ising model on MNs is investigated using a method typical for the SG theory, namely the replica method [34, 35]. Assuming purely FM interactions between all pairs of spins within all layers and using this method it is possible to evaluate critical temperature for the FM transition [7]; in the case of the Ising model on heterogeneous networks such critical temperature agrees quantitatively with that obtained from MC simulations. In Sec. 4.1 the class of models of MNs is defined for which the critical temperature is evaluated from the replica method. These models are based on the so-called static model [36, 37]: each layer of the MN is generated by first assigning a weight to each of N nodes and then connecting the nodes with edges taking into account the prescribed weights; weights associated with the same node can be different for different layers. FM and SG transitions in the Ising model on random ER graphs and heterogeneous networks obtained from the static model were investigated in Ref. [38] and [9], respectively, and the considerations in Sec. 4.2 and 4.3 below are a straightforward generalization of these studies to the case of MNs. In Sec. 4.4 and 4.5 results for the FM critical temperature are presented for various MNs obtained from the

static model and their generalization is proposed to the case of MNs with layers in the form of heterogeneous networks. In Sec. 4.6 results from the replica method are compared with those from the MF theory, Sec. 3, and with MC simulations.

4.1. The network models

Using the static model the (possibly heterogeneous) networks with a fixed number of nodes N and desired distributions of the degrees of nodes can be generated as follows [36, 37]. First, a weight v_i is assigned to each node so that the condition $\sum_{i=1}^N v_i = 1$ was fulfilled. Then, nodes are linked with edges in accordance with the prescribed sequence of weights, by selecting a pair of nodes i, j ($i \neq j$) with probabilities v_i, v_j , respectively, linking them with an edge and repeating this process $NK/2$ times. In this way, a network is obtained with the probability that the nodes i, j are linked by an edge $f_{ij} \approx v_i v_j$, with the mean degree of nodes $\langle k \rangle = K$, and with the distribution of the degrees of nodes depending on the choice of the weights. In particular, random ER graph is obtained if $v_i = 1/N$ is assumed for all i . For a sequence $v_i = i^{-\mu}/\zeta_N(\mu)$ associated with the nodes $i = 1, 2, \dots, N$, where $0 < \mu < 1$ and $\zeta_N(\mu) \approx N^{1-\mu}/(1-\mu)$, SF network is obtained with the distribution of the degrees of nodes $p_k \propto k^{-\gamma}$, $\gamma = 1 + 1/\mu$. In an ensemble of networks generated from the static model the mean degree of a given node i is $\langle k_i \rangle = v_i$.

The MN with a fixed set of nodes and two layers $G^{(A)}, G^{(B)}$ is generated by associating weights $v_i^{(A)}, v_i^{(B)}$ with the nodes separately to generate each layer. In this way, the layers can have different distributions of the degrees of nodes $p_{k^{(A)}}, p_{k^{(B)}}$. Let us note that the numbering of nodes $i = 1, 2, \dots, N$ while generating each layer can be assumed the same or different. In the case of random ER layers this distinction is unimportant, however, in the case of SF layers it can introduce correlations between the two sequences of weights $v_i^{(A)}, v_i^{(B)}, i = 1, 2, \dots, N$, where now and henceforth i denotes the index of the node in a MN, common for all layers. In particular, a MN with independent layers is obtained by randomly and independently associating weights from the two appropriate sets of weights with the nodes and then linking them with edges according to the prescribed sequence of weights within each layer.

4.2. Evaluation of the free energy

In order to go beyond the MF approximation the thermodynamic potentials for the Ising model should be evaluated on a statistical ensemble of MNs

generated according to a given rule. Hence, the free energy is $-\beta F = [\ln Z]_{av}$, where Z is the partition function for the Ising model on a particular MN, and the average $[\cdot]_{av}$ is taken over all possible random realizations of a MN of a given kind (since all non-zero exchange integrals are assumed as $J_{ij}^{(A)} = J_{ij}^{(B)} = J > 0$, there is no usual averaging over the quenched disorder of the exchange interactions). In the framework of the replica method the free energy is formally evaluated as $-\beta F = \lim_{n \rightarrow 0} \{[Z^n]_{av} - 1\} / n$. The average of the n -th power of the partition function is

$$[Z^n]_{av} = \text{Tr}_{\{s^\alpha\}} \left[\exp \left(\beta J \sum_{(i,j) \in G^{(A)}} \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) \exp \left(\beta J \sum_{(i,j) \in G^{(B)}} \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) \right]_{av} \quad (23)$$

i.e., it is the average of a product of n partition functions for non-interacting replicas (copies) of the system, the trace $\text{Tr}_{\{s^\alpha\}}$ is taken over all replicated spins $s_i^\alpha = \pm 1$, and $\alpha = 1, 2 \dots n$ is the replica index.

It should be emphasised that the generation of a MN takes place in two stages: first, in which the weights $v_i^{(A)}$, $v_i^{(B)}$ are separately assigned to the nodes $i = 1, 2, \dots N$, and second, in which the nodes are connected with edges taking into account the prescribed weights within each layer. Thus, in principle, the average in Eq. (23) should be taken over all possible realizations of the two above-mentioned independent random processes. However, at the first stage the two sequences of weights $v_i^{(A)}$, $v_i^{(B)}$, $i = 1, 2, \dots N$ can be assigned to the nodes independently, or certain correlations between them can be present. Then, the degrees of nodes $k_i^{(A)}$, $k_i^{(B)}$ within each layer can be also, on average, independent or correlated which can significantly affect the critical temperature for the FM transition, and averaging over all pairs of sequences of weights in Eq. (23) would hide the latter dependence. An important role of correlations between the degrees of nodes within layers of MNs was also emphasised in other phenomena, as percolation and cascading failures [21, 22, 39]. Thus, it is reasonable to consider separately classes of MNs characterized by given pairs of sequences of weights $v_i^{(A)}$, $v_i^{(B)}$, $i = 1, 2, \dots N$. Then the average in Eq. (23) is evaluated separately for each class, and is taken over all possible realizations of the two layers by connecting the nodes with edges according to the weights $v_i^{(A)}$, $v_i^{(B)}$, $i = 1, 2, \dots N$ characterizing this class. If necessary, a sort of further averaging over different classes of MNs (e.g., over all classes with the same correlation coefficient between the two sequences of weights $v_i^{(A)}$, $v_i^{(B)}$, $i = 1, 2, \dots N$) can be performed by

replacing the sums over N nodes by their expected values in the resulting expressions for the critical temperature.

For a class of MNs with fixed assignment of the weights $v_i^{(A)}, v_i^{(B)}$ to the nodes, the average in Eq. (23) can be taken independently over all realizations of the layers $G^{(A)}$ and $G^{(B)}$ in accordance with these weights. Denoting the respective averages by $[\cdot]_{av}^{(A)}, [\cdot]_{av}^{(B)}$ it is obtained that

$$[Z^n]_{av} = \text{Tr}_{\{s^\alpha\}} \left[\exp \left(\beta J \sum_{(i,j) \in G^{(A)}} \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) \right]_{av}^{(A)} \left[\exp \left(\beta J \sum_{(i,j) \in G^{(B)}} \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) \right]_{av}^{(B)}. \quad (24)$$

The two factors can be evaluated as in Ref. [9],

$$\begin{aligned} & \left[\exp \left(\beta J \sum_{(i,j) \in G^{(A)}} \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) \right]_{av}^{(A)} = \\ & \prod_{i < j} \left[(1 - f_{ij}^{(A)}) + f_{ij}^{(A)} \exp \left(\beta J \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) \right] = \\ & \exp \left\{ \sum_{i < j} \ln \left[1 + f_{ij}^{(A)} \left(\exp \left(\beta J \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) - 1 \right) \right] \right\} \approx \\ & \exp \left[\sum_{i < j} N K^{(A)} v_i^{(A)} v_j^{(A)} \left(\exp \left(\beta J \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) - 1 \right) \right], \quad (25) \end{aligned}$$

and similarly for the average $[\cdot]_{av}^{(B)}$. Then, since $s_i^\alpha s_j^\alpha = \pm 1$, the relation

$$\exp \left(\beta J \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) = \prod_{\alpha} \cosh \beta J (1 + s_i^\alpha s_j^\alpha \tanh \beta J) \quad (26)$$

can be used in Eq. (25), which yields

$$\begin{aligned} & \left[\exp \left(\beta J \sum_{(i,j) \in G^{(A)}} \sum_{\alpha=1}^n s_i^\alpha s_j^\alpha \right) \right]_{av}^{(A)} \propto \\ & \exp \left[\sum_{i < j} N K^{(A)} v_i^{(A)} v_j^{(A)} \left(\mathbf{T}_1 \sum_{\alpha} s_i^\alpha s_j^\alpha + \mathbf{T}_2 \sum_{\alpha < \beta} s_i^\alpha s_i^\beta s_j^\alpha s_j^\beta + \dots \right) \right], \quad (27) \end{aligned}$$

where $\mathbf{T}_1 = \cosh^n \beta J \tanh \beta J$, $\mathbf{T}_2 = \cosh^n \beta J \tanh^2 \beta J$, etc; similar expansion can be obtained for the average $[\cdot]_{av}^{(B)}$. Finally, after applying the

Hubbard-Stratonovich identity to the expressions of the form (27), separately for the two averages $[\cdot]_{av}^{(A)}$, $[\cdot]_{av}^{(B)}$, and grouping terms connected with the same nodes i it is obtained that

$$\begin{aligned} [Z^n]_{av} &= \int dq_{\alpha}^{(A)} \int dq_{\alpha\beta}^{(A)} \dots \int dq_{\alpha}^{(B)} \int dq_{\alpha\beta}^{(B)} \dots \exp \left[-Nn\beta f \left(q_{\alpha}^{(A)}, q_{\alpha\beta}^{(A)}, \dots, q_{\alpha}^{(B)}, q_{\alpha\beta}^{(B)} \dots \right) \right] \\ &\equiv \int d\mathbf{q} \exp \left[-Nn\beta f(\mathbf{q}) \right], \end{aligned} \quad (28)$$

with

$$\begin{aligned} n\beta f(\mathbf{q}) &= \frac{K^{(A)}\mathbf{T}_1}{2} \sum_{\alpha} q_{\alpha}^{(A)2} + \frac{K^{(B)}\mathbf{T}_1}{2} \sum_{\alpha} q_{\alpha}^{(B)2} + \dots \\ &+ \frac{K^{(A)}\mathbf{T}_2}{2} \sum_{\alpha < \beta} q_{\alpha\beta}^{(A)2} + \frac{K^{(B)}\mathbf{T}_2}{2} \sum_{\alpha < \beta} q_{\alpha\beta}^{(B)2} + \dots \\ &- \frac{1}{N} \sum_i \ln \text{Tr}_{\{s_i^{\alpha}\}} \exp \left(X_i^{(A)} + X_i^{(B)} \right), \end{aligned} \quad (29)$$

where $\text{Tr}_{\{s_i^{\alpha}\}}$ is the trace over the replicated spins at node i , and

$$X_i^{(A)} = NK^{(A)}\mathbf{T}_1 v_i^{(A)} \sum_{\alpha} q_{\alpha}^{(A)} s_i^{\alpha} + NK^{(A)}\mathbf{T}_2 v_i^{(A)} \sum_{\alpha < \beta} q_{\alpha\beta}^{(A)} s_i^{\alpha} s_i^{\beta} + \dots, \quad (30)$$

and similarly for $X_i^{(B)}$.

The elements of a set $\{\mathbf{q}\}$, $q_{\alpha}^{(A)}, q_{\alpha\beta}^{(A)}, \dots, q_{\alpha}^{(B)}, q_{\alpha\beta}^{(B)}, \dots$ form in a natural way two subsets of an infinite set of the order parameters associated with the two layers of the MN $G^{(A)}$, $G^{(B)}$. The first two order parameters, called magnetizations for convenience,

$$q_{\alpha}^{(A)} = \sum_i v_i^{(A)} \overline{s_i^{\alpha}}, \quad q_{\alpha}^{(B)} = \sum_i v_i^{(B)} \overline{s_i^{\alpha}}, \quad (31)$$

where the averages are evaluated as

$$\overline{s_i^{\alpha}} = \frac{\text{Tr}_{\{s_i^{\alpha}\}} s_i^{\alpha} \exp \left(X_i^{(A)} + X_i^{(B)} \right)}{\text{Tr}_{\{s_i^{\alpha}\}} \exp \left(X_i^{(A)} + X_i^{(B)} \right)},$$

resemble the "partial" order parameters $\langle S^{(A)} \rangle$, $\langle S^{(B)} \rangle$ of the MF approach in Sec. 3, with the average spin values weighted by $v_i^{(A)}$, $v_i^{(B)}$ rather than directly by the degrees of nodes $k_i^{(A)}$, $k_i^{(B)}$ within each layer. All remaining order parameters are unimportant for the evaluation of the critical temperature for the FM Ising model.

4.3. The replica symmetric free energy

The simplest replica symmetric (RS) solution for the order parameters is obtained under the assumption that spins with different replica index are indistinguishable. Thus in the case of the Ising model on a heterogeneous network the solution in the form $q_\alpha = m$, $q_{\alpha\beta} = q$, etc., for $\alpha, \beta = 1, 2 \dots n$, etc., is looked for [9]. However, in the case of a MN in general different weights $v_i^{(A)}$, $v_i^{(B)}$, $i = 1, 2 \dots N$ occur in the sums over the indices of nodes, Eq. (31), defining the two sets of the order parameters $q_\alpha^{(A)}$, $q_{\alpha\beta}^{(A)}$, \dots and $q_\alpha^{(B)}$, $q_{\alpha\beta}^{(B)}$, \dots . Thus, it is more natural to look for the solution in the form $q_\alpha^{(A)} = m^{(A)}$, $q_{\alpha\beta}^{(A)} = q^{(A)}$, etc., and $q_\alpha^{(B)} = m^{(B)}$, $q_{\alpha\beta}^{(B)} = q^{(B)}$, etc., for $\alpha, \beta = 1, 2 \dots n$, etc., where, in general, $m^{(A)} \neq m^{(B)}$, $q^{(A)} \neq q^{(B)}$, etc. It should be emphasised that this assumption has nothing to do with breaking the RS and follows simply from the existence of the two sets of the order parameters, associated with the layers $G^{(A)}$, $G^{(B)}$ of the MN.

Since at the present stage of research we are only interested in the evaluation of the critical temperature for the Ising model with purely FM interactions it is enough to keep only the terms containing magnetizations in the free energy, Eq. (29), and truncate it at the order m^2 . Assuming the above-mentioned form of the RS solution it is obtained that

$$n\beta f(m^{(A)}, m^{(B)}) = \frac{K^{(A)}\mathbf{T}_1}{2}nm^{(A)2} + \frac{K^{(B)}\mathbf{T}_1}{2}nm^{(B)2} + \dots - \frac{1}{N} \sum_i \ln \mathcal{Z}_i, \quad (32)$$

where

$$\mathcal{Z}_i = \text{Tr}_{\{s_i^\alpha\}} \exp \left(\eta_i \sum_\alpha s_i^\alpha + \dots \right) \approx (2 \cosh \eta_i)^n, \quad (33)$$

with

$$\eta_i = N\mathbf{T}_1 \left(K^{(A)}v_i^{(A)}m^{(A)} + K^{(B)}v_i^{(B)}m^{(B)} \right). \quad (34)$$

Thus the important part of the free energy is

$$\beta f(m^{(A)}, m^{(B)}) = \frac{K^{(A)}\mathbf{T}_1}{2}m^{(A)2} + \frac{K^{(B)}\mathbf{T}_1}{2}m^{(B)2} - \frac{1}{N} \sum_i \ln (2 \cosh \eta_i) \quad (35)$$

with $\mathbf{T}_1 = \tanh \beta$ in the limit $n \rightarrow 0$.

With $\beta f(\mathbf{q})$ given by Eq. (29) the integral in Eq. (28) can be evaluated using the saddle point method. For this purpose, assuming the RS solution,

the minimum of the function $f(m^{(A)}, m^{(B)})$ should be found, and the necessary condition for the existence of the extremum leads to the following set of self-consistent equations for the magnetizations $m^{(A)}, m^{(B)}$,

$$\frac{\partial f}{\partial m^{(A)}} = 0, \quad \frac{\partial f}{\partial m^{(B)}} = 0. \quad (36)$$

4.4. General equations for the critical temperature

For small $m^{(A)}, m^{(B)}$ after expanding the logarithm in Eq. (35) the free energy can be written as

$$\begin{aligned} \beta f(m^{(A)}, m^{(B)}) = \\ \frac{K^{(A)}\mathbf{T}_1}{2}m^{(A)2} + \frac{K^{(B)}\mathbf{T}_1}{2}m^{(B)2} - \frac{1}{2}N \sum_{i=1}^N \mathbf{T}_1^2 \left(K^{(A)}v_i^{(A)}m^{(A)} + K^{(B)}v_i^{(B)}m^{(B)} \right)^2. \end{aligned} \quad (37)$$

Then the system of Eq. (36) leads to the following system of linear equations for $m^{(A)}, m^{(B)}$,

$$\begin{aligned} \left(1 - NK^{(A)}\mathbf{T}_1 \sum_{i=1}^N v_i^{(A)2} \right) m^{(A)} - NK^{(B)}\mathbf{T}_1 \left(\sum_{i=1}^N v_i^{(A)}v_i^{(B)} \right) m^{(B)} &= 0 \\ -NK^{(A)}\mathbf{T}_1 \left(\sum_{i=1}^N v_i^{(A)}v_i^{(B)} \right) m^{(A)} + \left(1 - NK^{(B)}\mathbf{T}_1 \sum_{i=1}^N v_i^{(B)2} \right) m^{(B)} &= 0. \end{aligned} \quad (38)$$

Non-zero solutions of the system of Eq. (38) exist if the determinant is zero; from this condition the critical temperature for the FM transition from the RS solution, denoted as T_c^{RS} , can be obtained. It can be seen that the critical temperature depends on the correlation $\sum_{i=1}^N v_i^{(A)}v_i^{(B)}$ between the two sequences of weights $v_i^{(A)}, v_i^{(B)}$, $i = 1, 2, \dots, N$, assigned to the nodes during the generation of the two layers of the MN. In the next section the critical temperature is evaluated directly for several cases, e.g., for the Ising model on MNs with layers in the form of random ER graphs or SF networks.

4.5. Special cases

4.5.1. Random Erdős-Rényi layers

If the two layers of the MN are random ER graphs with mean degrees of nodes $K^{(A)}, K^{(B)}$ all weights are equal, $v_i^{(A)} = v_i^{(B)} = 1/N$, and the critical

temperature for the FM transition is

$$T_c^{RS} = J \tanh^{-1} \left(\frac{1}{K^{(A)} + K^{(B)}} \right). \quad (39)$$

This is of course the critical temperature for the FM transition in the Ising model on a random ER graph with the mean degree of nodes $K^{(A)} + K^{(B)}$. For large $K^{(A)}$, $K^{(B)}$ this result agrees with that from the MF approximation, Eq. (18), and is obvious since, as mentioned in Sec. 3.4.1, the two-layer MN is equivalent to a random ER graph with the mean degree of nodes equal to the sum of mean degrees of nodes within each layer.

4.5.2. Independent scale-free layers

Let us consider the case when the two layers of the MN are SF networks obtained from the static models with parameters $\mu^{(A)}$, $\mu^{(B)}$, with the distributions of the degrees of nodes obeying power scaling laws, $p_{k^{(A)}} \propto k_A^{-\gamma^{(A)}}$, $\gamma^{(A)} = 1 + 1/\mu^{(A)}$, $p_{k^{(B)}} \propto k_B^{-\gamma^{(B)}}$, $\gamma^{(B)} = 1 + 1/\mu^{(B)}$. First let us focus on the case when the layers are generated independently. To generate the layer $G^{(A)}$, the weights $v_i^{(A)}$ are randomly assigned to the nodes from the set of weights $v_j = j^{-\mu^{(A)}}/\zeta_N(\mu^{(A)})$, $j = 1, 2 \dots N$, and then the nodes are connected with $NK^{(A)}/2$ edges in accordance with the prescribed sequence of weights $v_i^{(A)}$. Next, the same procedure is repeated for the layer $G^{(B)}$, with the weights $v_i^{(B)}$ randomly assigned to the nodes from the set of weights $v_l = l^{-\mu^{(B)}}/\zeta_N(\mu^{(B)})$, $l = 1, 2 \dots N$. This means that the two sequences of weights $v_i^{(A)}$, $v_i^{(B)}$, $i = 1, 2, \dots N$ are independent. As a result, in Eq. (38) the sum over the products of weights can be approximated by its expected value,

$$\begin{aligned} N \sum_{i=1}^N v_i^{(A)} v_i^{(B)} &\approx N \langle \sum_{i=1}^N v_i^{(A)} v_i^{(B)} \rangle = N \sum_{i=1}^N \langle v_i^{(A)} v_i^{(B)} \rangle = N \sum_{i=1}^N \sum_{j=1, l=1}^N \frac{1}{N^2} v_j v_l \\ &= N \frac{1}{N^2} \sum_{i=1}^N \left(\sum_{j=1}^N v_j \right) \left(\sum_{l=1}^N v_l \right) = 1. \end{aligned} \quad (40)$$

This approximation is valid in typical cases of MNs with independently generated layers, and is applied instead of averaging the partition function in Eq. (23) over a class of MNs with mutually independent sequences of weights assigned to nodes when generating different layers. Besides, for $\mu^{(A)} < 1/2$

($\gamma^{(A)} > 3$), $\mu^{(B)} < 1/2$ ($\gamma^{(B)} > 3$) there is

$$N \sum_{i=1}^N v_i^{(A)2} = N \sum_{j=1}^N v_j^2 \approx \frac{(1 - \mu^{(A)})^2}{1 - 2\mu^{(A)}} = \frac{(\gamma^{(A)} - 2)^2}{(\gamma^{(A)} - 1)(\gamma^{(A)} - 3)}, \quad (41)$$

and similarly for $\sum_{i=1}^N v_i^{(B)2}$. Then the determinant of Eq. (38) is zero if

$$K^{(A)} K^{(B)} \left[\frac{(1 - \mu^{(A)})^2}{1 - 2\mu^{(A)}} \frac{(1 - \mu^{(B)})^2}{1 - 2\mu^{(B)}} - 1 \right] \mathbf{T}_1^2 - \left[K^{(A)} \frac{(1 - \mu^{(A)})^2}{1 - 2\mu^{(A)}} + K^{(B)} \frac{(1 - \mu^{(B)})^2}{1 - 2\mu^{(B)}} \right] \mathbf{T}_1 + 1 = 0. \quad (42)$$

As in the case of the critical temperature for the FM transition obtained in the MF approximation (Sec. 3.3) Eq. (42) can have two solutions. It is thus assumed that the higher of the two values of temperature fulfilling Eq. (42) corresponds to the critical temperature for the FM transition. Hence, the critical temperature is

$$T_c^{RS} = J \operatorname{atanh}^{-1} \left\{ \frac{K^{(A)} \frac{(1 - \mu^{(A)})^2}{1 - 2\mu^{(A)}} + K^{(B)} \frac{(1 - \mu^{(B)})^2}{1 - 2\mu^{(B)}} - \sqrt{\Delta}}{2K^{(A)} K^{(B)} \left[\frac{(1 - \mu^{(A)})^2}{1 - 2\mu^{(A)}} \frac{(1 - \mu^{(B)})^2}{1 - 2\mu^{(B)}} - 1 \right]} \right\}, \quad (43)$$

where

$$\Delta = \left[K^{(A)} \frac{(1 - \mu^{(A)})^2}{1 - 2\mu^{(A)}} - K^{(B)} \frac{(1 - \mu^{(B)})^2}{1 - 2\mu^{(B)}} \right]^2 + 4K^{(A)} K^{(B)}.$$

Thus, for the Ising model on MNs with layers in the form of SF networks with the distributions of the degrees of nodes obeying power scaling laws with the exponents $\gamma^{(A)} = (1 - \mu^{(A)})^{-1} > 3$, $\gamma^{(B)} = (1 - \mu^{(B)})^{-1} > 3$, the replica approach predicts a finite value of the critical temperature for the FM transition in the thermodynamic limit. In contrast, if $\mu^{(A)} > 1/2$ ($\gamma^{(A)} < 3$) or $\mu^{(B)} > 1/2$ ($\gamma^{(B)} < 3$) the corresponding sums in Eq. (41) diverge, and the critical temperature also diverges, so that in the thermodynamic limit the system remains in the FM phase at any temperature, as predicted using the MF approximation, too (Sec. 3.4.2).

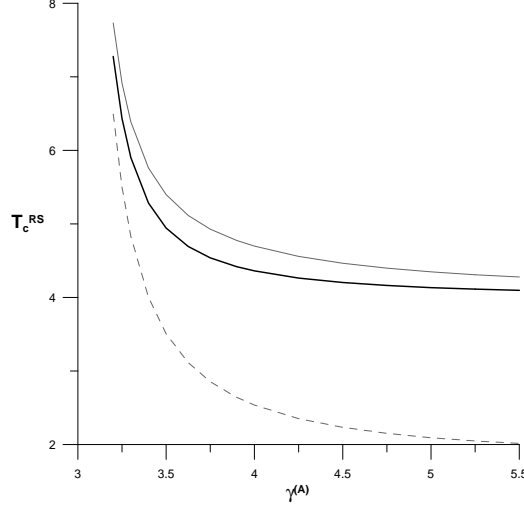


Figure 2: Critical temperature T_c^{RS} vs. $\gamma^{(A)}$ from the RS solution for the MN with SF layers with $K^{(A)} = K^{(B)} = 2$, $\gamma^{(B)} = 5.5$ and independent layers (black solid line), layers with maximum (gray solid line) and minimum (gray dashed line) correlation between sequences of weights.

4.5.3. Mutually correlated scale-free layers

So far the focus has been on the case of the Ising model on MNs with independently generated layers. However, if the layers are generated from the static model it is easy to introduce correlations between the degrees of nodes in different layers by associating appropriately the two sets of weights with the nodes; thus, here the effect of such correlations on the FM transition is briefly discussed.

In the case of two SF layers maximum correlation between the two sequences of weights, and thus between mean degrees of each node $\langle k_i^{(A)} \rangle$, $\langle k_i^{(B)} \rangle$ in the two layers in the ensemble of generated layers, is achieved by fixing the numbering of nodes $i = 1, 2 \dots N$ and associating the weights $v_i^{(A)} = i^{-\mu^{(A)}} / \zeta_N(\mu^{(A)})$, $v_i^{(B)} = i^{-\mu^{(B)}} / \zeta_N(\mu^{(B)})$ with the node with index i . As a result, the nodes which have high degree within one layer, have also, on average, high degree in the other layer and vice versa. Then for $\mu^{(A)} < 1/2$ ($\gamma^{(A)} > 3$), $\mu^{(B)} < 1/2$ ($\gamma^{(B)} > 3$)

$$N \sum_{i=1}^N v_i^{(A)} v_i^{(B)} \approx \frac{(1 - \mu^{(A)}) (1 - \mu^{(B)})}{1 - (\mu^{(A)} + \mu^{(B)})} \quad (44)$$

in Eq. (38). Since all MNs generated in this way are equivalent up to the

permutation of the indices of nodes, no further averaging as in Eq. (40) is necessary, and the critical temperature for the MF transition can be obtained by inserting Eq. (44) into Eq. (38) and solving the appropriately modified Eq. (42); the result is omitted for the sake of brevity. The critical temperature in this case is higher than that in the case of independently generated SF layers (Fig. 2).

In contrast, minimum correlation between the two sequences of weights is obtained if, for fixed numbering of nodes $i = 1, 2 \dots N$, the weights in the two layers are assumed as $v_i = i^{-\mu^{(A)}} / \zeta_N(\mu^{(A)})$, $v_{N-i}^{(B)} = i^{-\mu^{(B)}} / \zeta_N(\mu^{(B)})$. As a result, the nodes which have high degree within one layer have, on average, low degree in the other layer and $N \sum_{i=1}^N v_i^{(A)} v_i^{(B)} \xrightarrow{N \rightarrow \infty} 0$. After inserting this result in Eq. (38) it can be seen that in the thermodynamic limit the Ising model on the MN is decomposed into two practically non-interacting systems on separate SF networks corresponding to the two layers $G^{(A)}$, $G^{(B)}$, with their own critical temperatures,

$$T_c^{(A)} = J \operatorname{atanh}^{-1} \left\{ \frac{(\gamma^{(A)} - 1)(\gamma^{(A)} - 3)}{(\gamma^{(A)} - 2)^2} \right\},$$

and $T^{(B)}$ resulting from a similar formula. The critical temperature for the FM transition for the whole system is $T_c^{RS} = \max \{T_c^{(A)}, T_c^{(B)}\}$ and is lower than that in the case of independently generated SF layers (Fig. 2).

Thus, the critical temperature of the Ising model on a MN with SF layers generated from the static model depends on the correlation between the weights of nodes in different layers. However, in most cases T_c^{RS} should be close to that for the Ising model on a MN with independent SF layers, Eq. (43).

4.5.4. General heterogeneous layers

In networks generated from the static model there is $\langle k \rangle = K$, $N \sum_i v_i^2 = (\langle k^2 \rangle - \langle k \rangle^2) / \langle k \rangle^2$ [37]. Thus the result of Eq. (43) can be written in a more general form,

$$T_c^{RS} = J \operatorname{atanh}^{-1} \left\{ \frac{\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} + \frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} - 2 - \sqrt{\Delta}}{2 \left[\left(\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} - 1 \right) \left(\frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} - 1 \right) - \langle k^{(A)} \rangle \langle k^{(B)} \rangle \right]} \right\} \quad (45)$$

where

$$\Delta = \left(\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} - \frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} \right)^2 + 4 \langle k^{(A)} \rangle \langle k^{(B)} \rangle,$$

using the moments of the distributions of the degrees of nodes within each layer. It can be expected that Eq. (45) is valid for any multiplex network consisting of independently generated, possibly heterogeneous layers with finite second moments of the distributions of the degrees of nodes. This case corresponds to that considered in the framework of the MF theory in Sec. 3, where the joint distributions of the degrees of nodes in the MN were products of distributions within each layer, $p_{k^{(A)}, k^{(B)}} = p_{k^{(A)}} p_{k^{(B)}}$.

It should be noted that the necessary condition for the occurrence of the FM transition is that the critical temperature T_c^{RS} given by Eq. (45) is real and positive. It can be easily shown that this requires that

$$\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} + \frac{\langle k^{(B)2} \rangle}{\langle k^{(B)} \rangle} + \sqrt{\Delta} > 4, \quad (46)$$

which is also a condition for the occurrence of a giant component in a MN with two independently generated layers [17], in which nodes are connected via edges in any layer (but not necessarily in both layers). Thus, the FM transition can appear in MNs above the percolation threshold, in analogy with the case of complex networks [9].

As a particular case of Eq. (45) let us consider the critical temperature for the Ising model on a MN consisting of two layers with identical distributions of the degrees of nodes, i.e., $p_{k^{(A)}} = p_{k^{(B)}} = p_k$, and thus $\langle k^{(A)} \rangle = \langle k^{(B)} \rangle = \langle k \rangle$, $\langle k^{(A)2} \rangle = \langle k^{(B)2} \rangle = \langle k^2 \rangle$. Then

$$T_c^{RS} = J \operatorname{atanh}^{-1} \left[\left(\frac{\langle k^2 \rangle}{\langle k \rangle} + \langle k \rangle - 1 \right)^{-1} \right] = -2J \ln^{-1} \left(1 - \frac{2}{\frac{\langle k^2 \rangle}{\langle k \rangle} + \langle k \rangle} \right). \quad (47)$$

For large $\langle k^2 \rangle$, $\langle k \rangle$ the critical temperature can be approximated as

$$T_c^{RS} \approx J \left(\frac{\langle k^2 \rangle}{\langle k \rangle} + \langle k \rangle \right), \quad (48)$$

which is the MF result, Eq. (17), as expected for layers with large mean degrees of nodes.

In particular, if the distributions of the degrees of nodes within each layer obey power scaling laws in the form $p_{k^{(A)}} = (\gamma^{(A)} - 1) \tilde{m}^{\gamma^{(A)} - 1} (k^{(A)})^{-\gamma^{(A)}}$ for $k^{(A)} > \tilde{m}$, $p_{k^{(B)}} = (\gamma^{(B)} - 1) \tilde{m}^{\gamma^{(B)} - 1} (k^{(B)})^{-\gamma^{(B)}}$ for $k^{(B)} > \tilde{m}$, the critical

temperature can be obtained by inserting in Eq. (45)

$$\frac{\langle k^{(A)2} \rangle}{\langle k^{(A)} \rangle} = \tilde{m} \frac{\gamma^{(A)} - 2}{\gamma^{(A)} - 3}, \quad \langle k^{(A)} \rangle = \tilde{m} \frac{\gamma^{(A)} - 1}{\gamma^{(A)} - 2}, \quad (49)$$

and similar expressions for the moments of $p_{k^{(B)}}$. The result can be compared with Eq. (19) obtained in the MF approximation (see Sec. 3.4.2).

4.6. Comparison with the mean field theory and Monte Carlo simulations

In this section the critical temperature for the FM transition in the Ising model on MNs with independent SF layers obtained from the RS solution is compared to the corresponding MF critical temperature, Eq. (19), and to that obtained from MC simulations of the Ising model on MNs with independent SF layers generated from the Configuration Model (Sec. 3.5). Since the layers are independent the appropriate formula for T_c^{RS} is given by Eq. (45) with the moments of the distributions of the degrees of nodes given by Eq. (49). In Fig. 1(c) T_c^{RS} is shown for the Ising model on MNs with SF layers with $J = 1$ and with fixed $\gamma^{(A)} > 3$, $\gamma^{(B)} > 3$ and different \tilde{m} and compared with T_c^{MF} and T_c^{MC} . Though Eq. (45) is only a heuristic generalization of Eq. (43) to the case of arbitrary heterogeneous layers T_c^{RS} shows better quantitative agreement with T_c^{MC} than the MF critical temperature T_c^{MF} . The discrepancy between the analytic result from the RS solution and the result of MC simulations can be probably again attributed mainly to the fact that in the calculations leading to Eq. (43) the distributions $p_{k^{(A)}}$, $p_{k^{(B)}}$ were assumed continuous (cf. Eq. (41)). Besides, T_c^{RS} shows the same linear dependence on \tilde{m} as T_c^{MF} , predicted by Eq. (19), and as T_c^{MC} .

In Fig. 1(d) T_c^{RS} is shown for the Ising model on MNs with SF layers with $J = 1$, fixed $\gamma^{(B)} = 5.5$ and high $\tilde{m} = 20$ and with different $\gamma^{(A)} > 3$ and compared with T_c^{MF} and T_c^{MC} . It can be seen that T_c^{RS} diverges as $\gamma^{(A)} \rightarrow 3$, as expected, and the MF critical temperature approaches that obtained from the RS solution. In the whole range of $\gamma^{(A)}$ the critical temperature T_c^{RS} shows better quantitative agreement with T_c^{MC} than the MF value T_c^{MF} .

The qualitative, and to some extent even quantitative agreement between T_c^{MF} and the more rigorously evaluated T_c^{RS} confirms the validity of the MF approximation in the investigation of the Ising model on MNs with high enough density of connections.

5. Summary and conclusions

In this paper the FM Ising model was investigated on, possibly heterogeneous, MNs with separately generated layers which can have different structural properties and distributions of the degrees of nodes. Critical temperatures for the FM transition were evaluated from the MF approximation and using the replica method, from the RS solution, in particular in the case of random ER and SF layers, and compared with results of MC simulations. In the case of independently generated layers the two above-mentioned analytic methods yield qualitatively similar results for MNs with high density of connections, and the critical temperature obtained from the RS solution shows better quantitative agreement with numerical results. Using the replica approach it was also shown that the critical temperature depends sensitively on the correlations between the degrees of nodes within different layers. Investigation of the Ising model on MNs both in the framework of MF approximation and the replica method requires using different "partial" order parameters or magnetizations, respectively, associated with different layers of the MN, thus the study of the properties of the phase transition is a more difficult task than in the case of the Ising model on complex (possibly heterogeneous) networks.

The study of such a basic and simple model as the Ising model on MNs makes possible observation of the effect of the multilayer structure of the network of interactions on the properties of phase transitions. For example, in the framework of the heterogeneous MF theory it is possible to distinguish between the Ising model on a MN and on a super-network being a superposition of the layers even if the FM exchange interactions within all layers are equal. In particular, for strongly heterogeneous SF layers the MF critical temperatures in the two above-mentioned cases can be different. However, it is interesting to note that when they can be compared with the critical temperature reliably determined from MC simulations (e.g., for the Ising model on MNs with random ER layers or with SF layers with $\gamma^{(A)} > 3.5$, $\gamma^{(B)} > 3.5$) they turn out to be equal or very close to each other. This observation is important from the practical point of view since it is not always possible to tell if a given empirical network is a MN (i.e., it consists of separately generated layers) or a usual network (generated as a whole). Nevertheless, within the MF approximation in both cases the location of the predicted critical point for the ordering phase transition can be practically equal.

The heterogeneous MF theory presented in this paper can be easily gen-

eralized to the case of the Ising model on partly overlapping MNs. Using similar approach it should be also possible to investigate phase transitions in other systems on MNs, in particular with heterogeneous layers. The replica method can be applied to study the possible SG transition in the Ising model on MNs with both FM and antiferromagnetic interactions. Another interesting problem is determination of the critical exponents for the FM transition, in particular in the case of layers with different structure and distributions of the degrees of nodes, either from the heterogeneous MF theory or from the replica approach; this task is more difficult than usually since different "partial" order parameters are associated with the layers. In the above-mentioned cases the results of this paper provide a starting point for the future research.

References

- [1] R. Albert and A.-L. Barabási, Statistical mechanics of complex networks, *Rev. Mod. Phys.* 74 (2002) 47.
- [2] A.-L. Barabási, *Network Science*, Cambridge University Press, Cambridge 2016.
- [3] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Critical phenomena in complex networks, *Rev. Mod. Phys.* 80 (2008) 1276.
- [4] A. Barrat, M. Barthlemy, and A. Vespignani, *Dynamical Processes on Complex Networks*, Cambridge University Press: Cambridge, 2008.
- [5] A.-L. Barabási, R. Albert, Emergence of Scaling in Random Networks, *Science* 286 (1999) 509-512.
- [6] G. Bianconi, Mean field solution of the Ising model on a BarabásiAlbert network, *Phys. Lett. A* 303 (2002) 166-168.
- [7] M. Leone, A. Vázquez, A. Vespignani, and R. Zecchina, Ferromagnetic ordering in graphs with arbitrary degree distribution, *Eur. Phys. J. B* 28 (2002) 191-197.
- [8] K. Suchecki and J. A. Hołyst, Ising model on two connected Barabási-Albert networks, *Phys. Rev. E* 74 (2006) 011122.
- [9] D.-H. Kim, G.J. Rodgers, B. Kahng, and D. Kim, Spin-glass phase transition on scale-free networks, *Phys. Rev. E* 71 (2005) 056115.

- [10] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Ising model on networks with an arbitrary distribution of connections, *Phys. Rev. E* 66 (2002) 016104.
- [11] S. Yoon, A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Belief-propagation algorithm and the Ising model on networks with arbitrary distributions of motifs, *Phys. Rev. E* 84 (2011) 041144 .
- [12] A. Aleksiejuk, J. A. Hołyst, and D. Stauffer, Ferromagnetic phase transition in Barabási-Albert networks, *Physica A* 310 (2002) 260.
- [13] C. P. Herrero, Ising model in scale-free networks: A Monte Carlo simulation, *Phys. Rev. E* 69 (2004) 067109.
- [14] J. Menche, A. Valleriani, and R. Lipowsky, Sequences of phase transitions in Ising models on correlated networks, *Phys. Rev. E* 83 (2011) 061129.
- [15] C. P. Herrero, Ising model in clustered scale-free networks, *Phys. Rev. E* 91 (2015) 052812.
- [16] S. Boccaletti, G. Bianconi, R. Criado, C. I. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal, Z. Wang, M. Zanin, The structure and dynamics of multilayer networks, *Phys. Rep.* 544 (2014) 1.
- [17] Kyu-Min Lee, Jung Yeol Kim, Sangchul Lee and K.-I. Goh Multiplex networks, in: *Networks of Networks: The Last Frontier of Complexity*, ed. G. D’Agostino, A. Scala, Springer 2014.
- [18] Kyu-Min Lee, Byungjoon Min, and Kwang-Il Goh, Towards real-world complexity: an introduction to multiplex networks, *Eur. Phys. J. B* 88 (2015) 48.
- [19] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley and S. Havlin, Catastrophic cascade of failures in interdependent networks, *Nature* 464 (2010) 1025.
- [20] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Avalanche Collapse of Interdependent Networks, *Phys. Rev. Lett.* 109 (2012) 248701.

- [21] Byungjoon Min, Su Do Yi, Kyu-Min Lee and K.-I. Goh, Network robustness of multiplex networks with interlayer degree correlations, *Phys. Rev. E* 89 (2014) 042811.
- [22] Fei Tan, Yongxiang Xia, Wenping Zhang and Xinyu Jin, Cascading failures of loads in interconnected networks under intentional attack, *EPL* 102 (2013) 28009.
- [23] S. Gómez, A. Díaz-Guilera, J. Gómez-Gardenes, C. J. Pérez-Vicente, Y. Moreno, and A. Arenas, Diffusion Dynamics on Multiplex Networks, *Phys. Rev. Lett.* 110 (2013) 028701.
- [24] A. Solé-Ribalta, M. De Domenico, N. E. Kouvaris, A. Díaz-Guilera, S. Gómez, and A. Arenas, Spectral properties of the Laplacian of multiplex networks, *Phys. Rev. E* 88 (2013) 032807
- [25] Qingchu Wu, Yijun Lou, Wenfang Zhu, Epidemic outbreak for an SIS model in multiplex networks with immunization, *Mathematical Biosciences* 277 (2016) 3846.
- [26] L. G. Alvarez Zuzek, C. Buono, L. A. Braunstein, Epidemic spreading and immunization strategy in multiplex networks, *Journal of Physics: Conference Series* 640 (2015) 012007.
- [27] S. Jang, J. S. Lee, S. Hwang and B. Kahng, Ashkin-Teller model and diverse opinion phase transitions on multiplex networks, *Phys. Rev. E* 92 (2015) 022110.
- [28] M. E. J. Newman, in *Handbook of Graphs and Networks: From the Genome to the Internet*, ed. S. Bornholdt and H. G. Schuster, Wiley - VCH, Berlin 2003, p. 35-68.
- [29] P. Erdős and A. Rényi, On random graphs, *Publicationes Mathematicae* 6 (1959) 290-297.
- [30] K. Suchecki and J. A. Hołyst, Bistable-monostable transition in the Ising model on two connected complex networks, *Phys. Rev. E* 80 (2009) 031110.

- [31] L. E. Brennan, I. S. Reed, W. Sollfrey, A comparison of average-likelihood and maximum-likelihood ratio tests for detecting radar targets of unknown Doppler frequency, IEEE Trans. on Information Theor. 14 (1968) 104-110.
- [32] Quang Huy Nguen, Ch. Robert, Series expansions for sums of independent Pareto random variables, Statistics & Risk Modeling 32 (2015) 49.
- [33] K. Binder, D. Heermann, Monte Carlo Simulation in Statistical Physics, Springer-Verlag, Berlin, 1997.
- [34] M. Mézard, G. Parisi, and M. A. Virasoro, Spin Glass Theory and Beyond, World Scientific, Singapore, 1987.
- [35] H. Nishimori, Statistical Physics of Spin Glasses and Information Theory, Clarendon Press, Oxford 2001.
- [36] K.-I. Goh, B. Kahng, and D. Kim, Universal Behavior of Load Distribution in Scale-Free Networks, Phys. Rev. Lett. 87 (2001) 278701.
- [37] D.-S. Lee, K.-I. Goh, B. Kahng, D. Kim, Evolution of scale-free random graphs: Potts model formulation, Nucl. Phys. B 696 (2004) 351380.
- [38] L. Viana and A. J. Bray, Phase diagrams for dilute spin glasses, J. Phys. C: Solid State Phys. 18 (1985) 3037-3051.
- [39] Jung Yeol Kim and K.-I. Goh, Coevolution and Correlated Multiplexity in Multiplex Networks, Phys. Rev. Lett. 111 (2013) 058702.